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How does quantum impenetrability affect Aharonov–Bohm scattering?

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Abstract. It is shown that different forms of quantum inpenetrability lead to different physical consequences. This should be kept in mind in analysing experimental data. The relativistic impenetrability conditions are considered and the corresponding relativistic Aharonov–Bohm cross-sections are obtained. The possibility of the AB effect occurring in simply-connected space regions is discussed.

1. Introduction

The Aharonov–Bohm (AB) effect [1] is sometimes defined as the quantum manifestation of inaccessible fields. To obtain inaccessibility, one surrounds a region with $\boldsymbol{E}, \boldsymbol{H} \neq 0$ by an impenetrable screen of the suitable geometrical form. It is the goal of the present treatment to analyse different forms of impenetrability conditions. We say that a surface S is impenetrable for incoming particles when a component of quantum probability current normal to S vanishes at S. The plan of our exposition is as follows. In section 2 we study different non-relativistic impenetrability conditions and find that they lead to different physical situations. This should be kept in mind when comparing with experimental data. In the appendices we demonstrate that the simply-connected nature of a region of space does not guarantee the absence of the AB effect in it. Definite criteria are given for this. In section 3 we consider different relativistic impenetrability conditions and evaluate the corresponding relativistic AB scattering cross sections.

2. Non-relativistic impenetrability conditions

2.1. *The impenetrable cylindrical solenoid*

Consider an infinite cylindrical solenoid embedded into the impenetrable cylinder C of radius R (figure 1). In the non-relativistic case the quantum probability current is determined by

$$j = \frac{\hbar}{2\mathrm{i}\mu} (\bar{\psi} \operatorname{grad} \psi - \psi \operatorname{grad} \bar{\psi}) - \frac{e}{\mu c} \boldsymbol{A} |\psi|^2.$$

For the case under consideration the single non-vanishing component of the vector

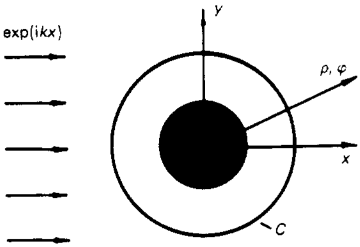

Figure 1. Shown is an infinite cylindrical solenoid (darkened) imbedded into the infinite cylinder C of radius R. On the surface of C the impenetrability BC is imposed. The arrows represent the wavevector of the incoming particles.

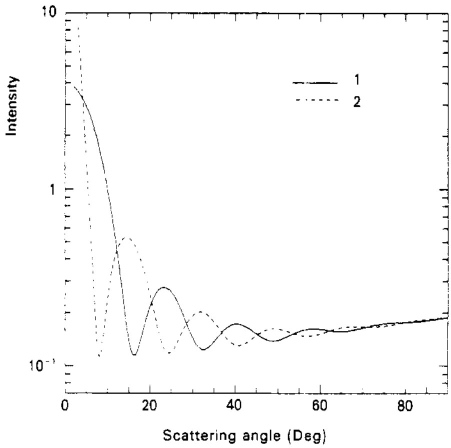

Figure 2. The non-relativistic intensity of particle scattering of the cylinder C for the case when the impenetrability BC (2.2) is imposed. The intensity is defined as the ratio of the scattering cross section (2.9) to the geometrical one ($=2R$). The intensity thus obtained is dimensionless. Curves 1 and 2 refer to the zero magnetic flux inside the solenoid and to $\gamma(\equiv e\varphi/hc)=\frac{1}{2}$, respectively. The parameter $kR=10$.

potential (VP) is $A_\varphi = \phi/2\pi\rho$ (ϕ is the magnetic field flux inside the solenoid). The aforementioned impenetrability condition ($j_\rho = 0$) reduces to

$$\psi\frac{\partial\bar\psi}{\partial\rho} - \bar\psi\frac{\partial\psi}{\partial\rho} = 0 \qquad \text{at } \rho = R. \tag{2.1}$$

Usually, one satisfies this condition simply by putting

$$\psi = 0 \qquad \text{at } \rho = R. \tag{2.2}$$

There are a variety of other ways to meet (2.1). Here are the two simplest

$$\frac{\partial\psi}{\partial\rho} = 0 \qquad \text{at } \rho = R \tag{2.3}$$

$$\frac{\partial\psi}{\partial\rho} = \alpha\psi \qquad \text{at } \rho = R \tag{2.4}$$

(α is an arbitrary real constant). In either case the wavefunction can be presented as

$$\psi = \psi_{\mathrm{AB}} + \psi_{\mathrm{S}}. \tag{2.5}$$

Here, ψ_{AB} is the scattering wavefunction for the infinitely thin ($kR \ll 1$) non-shielded solenoid

$$\psi_{\mathrm{AB}} = \sum \exp[\mathrm{i}\pi(|m| - \tfrac{1}{2}|m-\gamma|) + \mathrm{i}m\varphi]J_{|m-\gamma|}(k\rho) \tag{2.6}$$

($k^2 = 2\mu E/\hbar^2$, $\gamma = e\phi/hc$). From now on we will assume that a summation index, if not indicated otherwise, should range from $-\infty$ to ∞. The asymptotic form of ψ_{AB} which is valid for all scattering angles φ was first obtained in [2]:

$$\psi_{\mathrm{AB}} \approx \exp[\mathrm{i}\gamma(\varphi - \pi) + \mathrm{i}kx] + \mathrm{i}\sin\pi\gamma\frac{\exp(\mathrm{i}\varphi/2)\exp(\mathrm{i}k\rho)}{(1 - 2\pi\mathrm{i}k\rho\sin^2\varphi/2)^{1/2}}.$$

For angles φ not too small ($k\rho \sin^2 \varphi/2 \gg 1$) one recovers the asymptotic behaviour given in [1]

$$\psi_{AB} \approx \exp[ikx + i\gamma(\varphi - \pi)] + \frac{1}{\sqrt{\rho}} \exp(ik\rho) f_{AB}(\varphi)$$

$$f_{AB}(\varphi) = -\frac{1}{\sqrt{2\pi i k}} \frac{\exp(i\varphi/2)}{\sin \varphi/2} \sin \pi\gamma \tag{2.7}$$

$$\sigma_{AB}(\varphi) = \frac{1}{2\pi k} \frac{\sin^2 \pi\gamma}{\sin^2 \varphi/2}.$$

(For definiteness and without loss of generality, we choose $0 \leqslant \gamma \leqslant 1$). The second term in (2.5) takes into account the shielding of the solenoid:

$$\psi_S = \sum C_m \exp[i\pi(|m| - \tfrac{1}{2}|m - \gamma|) + im\varphi] H^{(1)}_{|m-\gamma|}(k\rho). \tag{2.8}$$

The coefficients C_m entering into (2.8) are determined by the boundary conditions (BC) at $\rho = R$

$$C_m = -J_{|m-\gamma|}/H^{(1)}_{|m-\gamma|} \qquad \text{for BC (2.2)}$$

$$C_m = -\dot{J}_{|m-\gamma|}/\dot{H}^{(1)}_{|m-\gamma|} \qquad \text{for BC (2.3)}$$

$$C_m = -\frac{k\dot{J}_{|m-\gamma|} - \alpha J_{|m-\gamma|}}{k\dot{H}^{(1)}_{|m-\gamma|} - \alpha H^{(1)}_{|m-\gamma|}} \qquad \text{for BC (2.4).}$$

From now on we do not indicate the arguments of the Bessel and Hankel functions if they are kR. The dot over these functions represents the derivative WRT their arguments. As $\rho \to \infty$ it follows from (2.8) that

$$\psi_S \sim \frac{1}{\sqrt{\rho}} \exp(ik\rho) f_S(\varphi)$$

$$f_S(\varphi) = \left(\frac{2}{\pi i k}\right)^{1/2} \sum C_m \exp[i\pi(|m| - |m - \gamma|) + im\varphi].$$

The total scattering amplitude and cross section are

$$f = f_{AB} + f_S \qquad \sigma = |f|^2. \tag{2.9}$$

We observe the explicit dependence of the scattering cross section on the concrete realization of the impenetrability BC (2.1). The typical cross sections shown in figures 2-4 demonstrate strong sensitivity to the particular choice of BC. Although BC (2.2)-(2.4) are trivial from the mathematical standpoint (they correspond to the Dirichlet, Neumann and mixed boundary problems, respectively), they are definitely not trivial from the physical viewpoint. In fact, (2.2)-(2.4) represent different definitions of quantum impenetrability. This should not be overlooked in the analysis of the experimental data. At this stage, we are not interested in the wavefunction behaviour inside C. It is determined by the particular form of the repulsive potential inside C. A reservation is needed. The presentation of the wavefunctions in the form (2.5), (2.6) and (2.8) implies single-valuedness of the wavefunctions and the BC corresponding to them. It is known [3] that the AB effect exists if only single-valued wavefunctions are used. Mathematically, the multivaluedness of the wavefunctions is not abandoned in multiply connected regions of space (the famous Pauli proof of the single-valuedness of the wavefunctions holds only in simply connected space regions). In fact, the recent

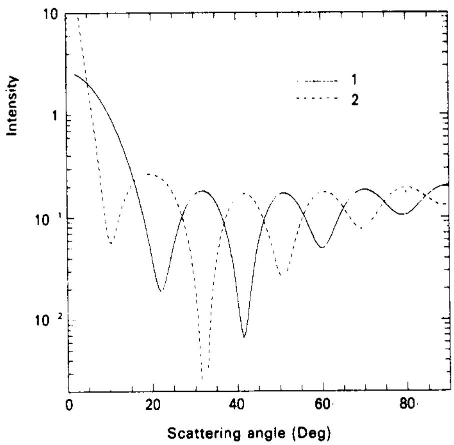

Figure 3. The same as in figure 2 but for the case when BC (2.3) is imposed.

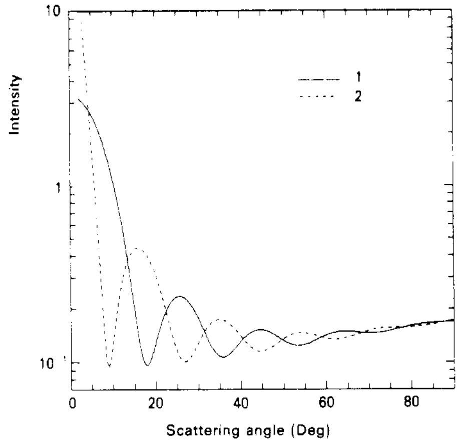

Figure 4. The same as in figure 2 for the case when BC (2.4) is imposed. The parameter $\alpha = 1$.

discussion on the existence of the AB effect is due to this theoretical ambiguity. Finally, we would like to mention two earlier papers [4] in which the influence of the change of boundary conditions on the scattering process was studied (irrespective of the AB effect).

2.2. *Scattering on the impenetrable sphere*

Consider now the impenetrable sphere S of radius R. In the absence of the magnetic field the quantum impenetrability condition is:

$$j_r = \frac{\hbar}{2i\mu}\left(\bar{\psi}\frac{\partial\psi}{\partial r} - \psi\frac{\partial\bar{\psi}}{\partial r}\right) = 0 \qquad \text{at } r = R. \tag{2.10}$$

The simplest solutions of these equations are as follows:

$$\psi = 0 \qquad \text{at } r = R \tag{2.11}$$

$$\frac{\partial\psi}{\partial r} = 0 \qquad \text{at } r = R \tag{2.12}$$

$$\frac{\partial\psi}{\partial r} = \alpha\psi \qquad \text{at } r = R \tag{2.13}$$

(α is real).

In either of these cases the wavefunction and scattering amplitude are given by

$$\begin{aligned} \psi_0 &= \exp(ikz) + \sqrt{\frac{\pi}{2kr}}\sum i^l(2l+1)C_l H^{(1)}_{l+1/2}(kr)P_l(\cos\theta) \\ f_S(\theta) &= \frac{1}{ik}\sum(2l+1)C_l P_l(\cos\theta) \\ \sigma &= |f_S(\theta)|^r \end{aligned} \tag{2.14}$$

where

$$C_l = -\frac{J_{l+1/2}}{H^{(1)}_{l+1/2}} \qquad \text{for BC (2.11)}$$

$$C_l = -\frac{(l+1)J_{l+1/2} - kRJ_{l-1/2}}{(l+1)H^{(1)}_{l+1/2} - kRH^{(1)}_{l-1/2}} \qquad \text{for BC (2.12)}$$

$$C_l = -\frac{(l+1+\alpha R)J_{l+1/2} - kRJ_{l-1/2}}{(l+1+\alpha R)H^{(1)}_{l+1/2} - kRH^{(1)}_{l-1/2}} \qquad \text{for BC (2.13).}$$

The corresponding cross sections ($\sigma = |f_S|^2$) are shown in figure 5. As for the impenetrable cylinder, the strong dependence on the particular realization of the impenetrability BC is observed.

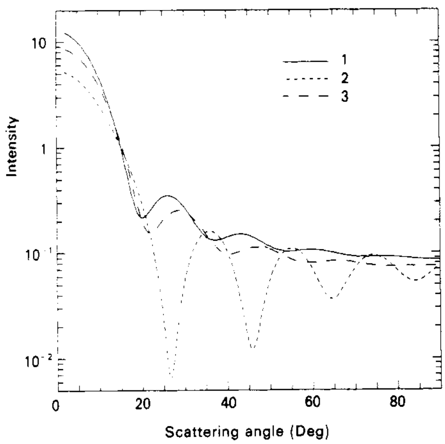

Figure 5. The non-relativistic intensity of particle scattering on the impenetrable sphere S of radius R. The intensity is defined as the ratio of the scattering cross section (2.14) to the geometrical one ($=\pi R^2$). The curves 1, 2 and 3 refer to the impenetrability BC (2.11), (2.12) and (2.13) (with $\alpha = 1$), respectively. The parameter $kR = 10$.

2.3. *Impenetrable sphere with a magnetic field inside it*

Install into the sphere S the toroidal solenoid $(\rho - d)^2 + z^2 = R_0^2$ (figure 6). The magnetic field differs from zero only inside the solenoid ($H_\rho = H_z = 0$, $H_\varphi = g/\rho$, $g = (1/2\pi)\phi(d - \sqrt{d^2 - R_0^2})^{-1}$, ϕ is the magnetic flux inside the solenoid). In the Coulomb gauge two non-vanishing components of the VP are A_r and A_θ [5]. At large distances they fall as r^{-3}

$$A_r \sim \frac{\pi g\, \mathrm{d}R_0^2}{2r^3}\cos\theta \qquad A_\theta \sim \frac{\pi g\, \mathrm{d}R_0^2}{4r^3}\sin\theta.$$

We present here their explicit expressions for the infinitely thin ($R_0 \ll d$) solenoid [5]:

$$A_\theta = \frac{R_0^2 g}{2(\mathrm{d}r\sin\theta)^{3/2}}\frac{1}{\mathrm{sh}\,\mu}(d\sin\theta Q^1_{-1/2}(\mathrm{ch}\,\mu) - rQ^1_{1/2}(\mathrm{ch}\,\mu))$$

$$A_r = -\frac{R_0^2 g d\cos\theta}{2(\mathrm{d}r\sin\theta)^{3/2}}\frac{1}{\mathrm{sh}\,\mu}Q^1_{-1/2}(\mathrm{ch}\,\mu). \qquad (2.15)$$

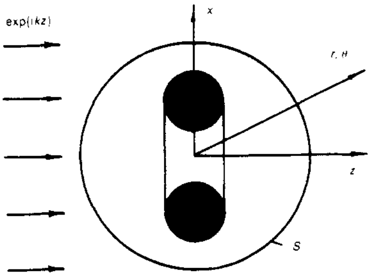

Figure 6. Shown is the toroidal solenoid (darkened) imbedded into the impenetrable sphere S. On its surface the impenetrability BC (2.16) is imposed.

(Here $\operatorname{ch}\mu = (r^2 + d^2)/2\,dr\sin\theta$, Q_v^σ is a Legendre function of the second kind.) As $A_r \neq 0$ outside the sphere S, the impenetrability condition is modified:

$$j_r = \frac{\hbar}{2\mathrm{i}\mu}\left(\bar{\psi}\frac{\partial\psi}{\partial r} - \psi\frac{\partial\bar{\psi}}{\partial r}\right) - \frac{e}{\mu c}A_r|\psi|^2 = 0 \qquad \text{for } r = R. \tag{2.16}$$

Now, the following question arises: does the impenetrability condition (2.16) guarantee the absence of observable effects associated with the non-vanishing of VP outside the sphere S? At first glance the answer is almost evident. Indeed, after the BC at $r = R$ is imposed, one operates in a simply connected space (the exterior of the space S). Usually, it is believed [6] that the observable effects are impossible in the simply connected region with $H = 0$ (as it is possible to make there a gauge transformation which eliminates the VP). The counter-examples presented in a very interesting paper [7], published in this journal, seem to cast doubt on this statement. Because of this, very particular case should be considered separately without appeal to the simple connectedness arguments. In [8] an ingenious gedanken experiment (with no reliance on the AB effect) was proposed, permitting one in principle to measure phaseshift effects in simply connected space regions. Thus the observability of physical effects in simply connected regions, having fundamental importance, deserves special consideration. As these questions are slightly out of the mainstream of the present exposition, we only sketch them here. All details may be found in the appendices. Mathematically, a space region R_{M} is multiply connected if there is a closed contour in it which cannot be contracted into a point without leaving R_{M}. For example, the space region outside the torus T is multiply connected (the contour passing through the torus hole cannot be contracted to a point). Now, surround T by the sphere S. The space region R_{S} lying outside S is simply connected (each contour outside S can be contracted to a point). The Schrödinger equation may be solved outside S if some BC is imposed on its surface. For arbitrary BC the particles in general penetrate (as $j_r \neq 0$) from R_{S} to the multiply connected region R_{M} lying between the torus T and the surface of S. The switching on of the magnetic field inside T may lead (for the same BC on the surface of S) to observable effects in the simply connected space region lying outside S even if the torus is impenetrable for particles. To avoid such observable effects, the gauge-invariant BC should be imposed on the surface of S. As the impenetrability BC (2.16) is gauge invariant, so it guarantees the absence of observable effects. The arising complications are discussed in the appendices.

3. Relativistic impenetrability conditions and the AB effect

3.1. Relativistic scattering on the cylindrical solenoid

Now we turn to the relativistic AB effect. To the best of our knowledge there are only a few references treating this subject. It was shown in [9] that for the infinitely thin solenoid the relativistic AB scattering cross section is $\sqrt{1-\beta^2}$ times the non-relativistic one. Relativistic scattering on the impenetrable cylindrical solenoid of finite radius was studied in [10]. The authors of [10] claim that it is impossible for all components of the Dirac wavefunctions to vanish at the surface of C. Relativistic scattering on a finite cylindrical potential barrier of a constant value was investigated in [11]. It was shown there that it is impossible to achieve continuity for all components of the Dirac wavefunction and their derivatives at the boundary of this barrier. In typical experiments [12] testing the AB effect, the electron energy is of the order 100 keV, which corresponds to $\beta(\equiv v/c)\approx 0.6$. This obliges us to analyse this situation more carefully.

Outside the cylinder C the wavefunction satisfies the Dirac equation

$$H\psi = \mathscr{E}\psi \qquad H = -\mathrm{i}\hbar c\boldsymbol{\alpha}\cdot\left(\boldsymbol{\nabla}-\frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)+\mu c^2\beta$$

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} \qquad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

We now develop ψ into states with definite projection of the angular momentum

$$J_3=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi}+\frac{1}{2}\hbar\Sigma_3 \qquad \boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$$

$$\psi=\sum\psi_m \qquad \psi_m=\begin{bmatrix}\psi_{1m}\\ \psi_{2m}\\ \psi_{3m}\\ \psi_{4m}\end{bmatrix} \qquad J_3\psi_m=\hbar(m+\tfrac{1}{2})\psi_m \tag{3.1}$$

$$\psi_{1m}=U_{1m}\exp(\mathrm{i}m\varphi) \qquad \psi_{2m}=U_{2m}\exp[\mathrm{i}(m+1)\varphi]$$

$$\psi_{3m}=U_{3m}\exp(\mathrm{i}m\varphi) \qquad \psi_{4m}=U_{4m}\exp[\mathrm{i}(m+1)\varphi].$$

It turns out that U_{1m} and U_{2m} are linear combinations of Bessel functions:

$$U_{1m}=A_m[J_{m-\gamma}(k\rho)+B_mH^{(1)}_{m-\gamma}(k\rho)]$$

$$U_{2m}=C_m[J_{m+1-\gamma}(k\rho)+D_mH^{(1)}_{m+1-\gamma}(k\rho)]$$

($k=\sqrt{\mathscr{E}^2-\mu^2c^4}/\hbar c$). Small components of the Dirac wavefunction are expressed through large U_{1m} and U_{2m}

$$\begin{aligned}U_{3m}&=-\mathrm{i}\eta\left(\frac{\mathrm{d}}{\mathrm{d}k\rho}+\frac{m+1-\gamma}{k\rho}\right)U_{2m}\\ &=-\mathrm{i}\eta C_m[J_{m-\gamma}(k\rho)+D_mH^{(1)}_{m-\gamma}(k\rho)]\\ U_{4m}&=-\mathrm{i}\eta\left(\frac{\mathrm{d}}{\mathrm{d}k\rho}-\frac{m-\gamma}{k\rho}\right)U_{1m}\\ &=-\mathrm{i}\eta A_m[J_{m+1-\gamma}(k\rho)+B_mH^{(1)}_{m+1-\gamma}(k\rho)]\end{aligned} \tag{3.2}$$

($\eta=((\mathscr{E}-\mu c^2)/(\mathscr{E}+\mu c^2))^{1/2}$). For definiteness we choose the incoming wave to be propagating in the positive x direction and having positive energy and helicity

$$\boldsymbol{\Sigma}\cdot\left(\boldsymbol{P}-\frac{e}{c}\boldsymbol{A}\right)\psi_{\mathrm{inc}}=\hbar k\,\psi_{\mathrm{inc}}.$$

This fixes both ψ_{inc} and the coefficients A_m and C_m

$$\psi_{\mathrm{inc}}=\exp[\mathrm{i}kx+\mathrm{i}\gamma(\varphi-\pi)]\,U_\eta$$

$$A_m=\exp\left(\mathrm{i}\pi\frac{m+\gamma}{2}\right)\qquad C_m=\exp\left(\mathrm{i}\pi\frac{m+\gamma+1}{2}\right)\qquad U_\eta=\begin{bmatrix}1\\1\\\eta\\\eta\end{bmatrix}.$$

We insert A_m and C_m into (3.1) and present ψ in the form

$$\psi=(\psi_{\mathrm{AB}}+\psi_{\mathrm{S}}^0)\,U_\eta+\psi_{\mathrm{S}}. \tag{3.3}$$

Here ψ_{AB} is determined by (2.6) and

$$\psi_{\mathrm{S}}^0=\mathrm{i}\sin\pi\gamma\sum_{m=0}^{-\infty}\exp\left(\mathrm{i}\pi\frac{\gamma-m}{2}\right)H_{\gamma-m}^{(1)}(k\rho)\exp(\mathrm{i}m\varphi)$$

$$\psi_{\mathrm{S}}=\begin{bmatrix}\psi_1^{\mathrm{S}}\\\psi_2^{\mathrm{S}}\\\psi_3^{\mathrm{S}}\\\psi_4^{\mathrm{S}}\end{bmatrix}$$

$$\psi_1^{\mathrm{S}}=\sum\exp\left(\mathrm{i}\pi\frac{m+\gamma}{2}\right)B_mH_{m-\gamma}^{(1)}(k\rho)\exp(\mathrm{i}m\varphi)$$

$$\psi_2^{\mathrm{S}}=\sum\exp\left(\mathrm{i}\pi\frac{m+\gamma}{2}\right)D_{m-1}H_{m-\gamma}^{(1)}(k\rho)\exp(\mathrm{i}m\varphi)$$

$$\psi_3^{\mathrm{S}}=\eta\sum\exp\left(\mathrm{i}\pi\frac{m+\gamma}{2}\right)D_mH_{m-\gamma}^{(1)}(k\rho)\exp(\mathrm{i}m\varphi)$$

$$\psi_4^{\mathrm{S}}=\eta\sum\exp\left(\mathrm{i}\pi\frac{m+\gamma}{2}\right)B_{m-1}H_{m-\gamma}^{(1)}(k\rho)\exp(\mathrm{i}m\varphi).$$

As $\rho\to\infty$ one gets the following asymptotic behaviour for ψ:

$$\psi\approx\exp[\mathrm{i}kx+\mathrm{i}\gamma(\varphi-\pi)]\,U_\eta+\frac{1}{\sqrt{\rho}}\exp(\mathrm{i}k\rho)f(\varphi) \tag{3.4}$$

where the spinorial scattering amplitude is

$$f=\begin{bmatrix}f_1\\f_2\\\eta f_2\exp(-\mathrm{i}\varphi)\\\eta f_1\exp(\mathrm{i}\varphi)\end{bmatrix}$$

$$f_1=\sqrt{\frac{2}{\pi\mathrm{i}k}}\exp(\mathrm{i}\pi\gamma)\sum B_m\exp(\mathrm{i}m\varphi) \tag{3.5}$$

$$f_2=\sqrt{\frac{2}{\pi\mathrm{i}k}}\exp(\mathrm{i}\pi\gamma)\sum D_{m-1}\exp(\mathrm{i}m\varphi).$$

The scattering cross section is

$$\sigma=\tfrac{1}{2}(|f_1|^2+|f_2|^2). \tag{3.6}$$

The coefficients B_m and D_m are fixed by the boundary condition at $\rho = R$. For example, we may require the large components of the Dirac wavefunction to vanish:

$$U_{1m} = U_{2m} = 0 \qquad D_{m-1} = B_m = -\frac{J_{m-\gamma}}{H^{(1)}_{m-\gamma}}. \tag{3.7}$$

Insert these values into (3.2) and obtain for $\rho = R$

$$|U_{3m}|^2 = \frac{4\eta^2}{\pi^2 k^2 R^2}(J^2_{m-\gamma} + Y^2_{m-\gamma})^{-1}$$

$$|U_{4m}|^2 = \frac{4\eta^2}{\pi^2 k^2 R^2}(J^2_{m+1-\gamma} + Y^2_{m+1-\gamma})^{-1}.$$

In the mentioned experiments on the AB effect [12] $kR \sim 10^6$. We replace Bessel and Neumann functions with their asymptotic values: $|U_{3m}|^2 \approx |U_{4m}|^2 \approx 2\eta^2/(\pi kR) \ll 1$. We see that although the non-vanishing of small components of the Dirac wavefunction indeed takes place [10], it is negligible for the existing experimental facilities. Now we insert (3.7) into (3.5) and (3.6):

$$f_1 = f_2 = f = -\sqrt{\frac{2}{\pi i k}} \exp(i\pi\gamma) \sum \frac{J_{m-\gamma}}{H^{(1)}_{m-\gamma}} \exp(im\varphi).$$

This sum contains Bessel functions with both positive and negative indices. Using the well known identities

$$J_{-\nu} = \exp(-i\pi\nu)J_\nu + i \sin \pi\nu \cdot H^{(1)}_\nu \qquad H^{(1)}_{-\nu} = \exp(i\pi\nu)H^{(1)}_\nu$$

we remove the latter and get

$$f = -\sqrt{\frac{2}{\pi i k}}\left(\tfrac{1}{2}\sin \pi\gamma \frac{\exp(i\varphi/2)}{\sin \varphi/2} + \sum \exp[i\pi(|m| - |m-\gamma|) + im\varphi]\frac{J_{|m-\gamma|}}{H^{(1)}_{|m-\gamma|}}\right). \tag{3.8}$$

Typical cross-sections ($\sigma = |f|^2$) are shown in figure 7. The amplitude (3.8) has the same form as the non-relativistic one (2.9). The only difference is in the meaning of

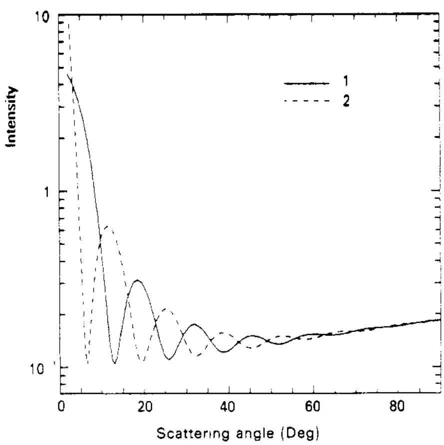

Figure 7. The relativistic intensity of the AB electron scattering on the impenetrable cylinder C of radius R for $\gamma = 0$ (curve 1) and $\gamma = \frac{1}{2}$ (curve 2). The impenetrability BC is chosen as the vanishing of large components of the Dirac wavefunction. The kinetic electron energy is 150 keV. The parameter $kR = 10$.

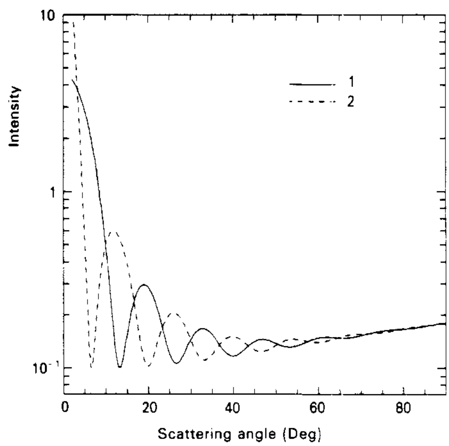

Figure 8. The same as in figure 7 but for the impenetrability condition (3.11).

the wavenumber k. The relativistic k occurring in (3.8) is expressed through the non-relativistic k occurring in (2.9) in the same way as for momenta: $k(3.8) = k(2.9)/\sqrt{1-\beta^2}$. This means that it is enough to substitute in (2.9) the relativistic k instead of non-relativistic one in order to obtain the relativistic amplitude (3.8). *A posteriori* this fact may be understood as follows: the particular components of the Dirac wavefunction in the region where $H = 0$ satisfies a second-order differential equation. It formally coincides with the non-relativistic Schrödinger equation if the above substitution of the wavenumbers is made. For the infinitely thin solenoid ($kR \ll 1$) the sum in (3.8) can be neglected and

$$f_{\mathrm{AB}}^{\mathrm{rel}} = -\frac{1}{\sqrt{2\pi i k}} \sin \pi\gamma \frac{\exp(\mathrm{i}\varphi/2)}{\sin \varphi/2} \qquad \sigma_{\mathrm{AB}}^{\mathrm{rel}} = \frac{1}{2\pi k} \frac{\sin^2 \pi\gamma}{\sin^2 \varphi/2}. \tag{3.9}$$

Due to the different meaning of k in (3.9) and in (2.7), $\sigma_{\mathrm{AB}}^{\mathrm{rel}} = \sqrt{1-\beta^2}\, \sigma_{\mathrm{AB}}^{\mathrm{nonrel}}$, which agrees with [9].

We see that the non-vanishing of small components of the Dirac wavefunction observed in [5] is due to the non-relativistic nature of the boundary condition at $\rho = R$. The impenetrability condition may be exactly fulfilled if one imposes the relativistic boundary condition. To be precise, we require the vanishing of the normal (to the surface of C) component of the Dirac probability current: $\boldsymbol{n} \cdot \boldsymbol{j}_{\mathrm{D}} = 0$ for $\rho = R$. Here $\boldsymbol{n}$ is the normal to the surface of C ($\boldsymbol{n} = (\cos \varphi, \sin \varphi, 0; 0)$) and $\boldsymbol{j}_{\mathrm{D}} = ec\psi^{+}\boldsymbol{\alpha}\psi$. As a result, one gets

$$\psi^{+}\boldsymbol{\alpha n}\psi = 0 \qquad \text{for } \rho = R. \tag{3.10}$$

(This relation is widely used in the MIT bag model, see, e.g., [13].) This quadratic expression is not convenient for practical use. The following relativistic boundary condition, which is linear in the Dirac wavefunction components, is also used in the MIT bag model [13]:

$$\mathrm{i}\boldsymbol{\alpha n}\psi = \beta\psi \qquad \text{for } \rho = R. \tag{3.11}$$

It is easy to see that (3.10) follows from (3.11). The reverse is not true. Equation (3.11) being applied to the Dirac wavefunction (3.1) results in: $U_{4m} = -\mathrm{i}U_{1m}$, $U_{3m} = -\mathrm{i}U_{2m}$ for $\rho = R$. Thus the coefficients B_m and D_m may be evaluated

$$B_m = -\frac{J_{m-\gamma} + \eta J_{m+1-\gamma}}{H^{(1)}_{m-\gamma} + \eta H^{(1)}_{m+1-\gamma}}$$

$$D_m = -\frac{J_{m+1-\gamma} - \eta J_{m-\gamma}}{H^{(1)}_{m+1-\gamma} - \eta H^{(1)}_{m-\gamma}}.$$

Substitute them into (3.5):

$$\begin{aligned} f_1 = -\sqrt{\frac{2}{\pi i k}} \Bigg(& \frac{1}{2} \sin \pi\gamma \frac{\exp(\mathrm{i}\varphi/2)}{\sin \varphi/2} + \exp(\mathrm{i}\pi\gamma) \sum_{m=1}^{\infty} \frac{J_{m-\gamma} + \eta J_{m+1-\gamma}}{H^{(1)}_{m-\gamma} + \eta H^{(1)}_{m+1-\gamma}} \exp(\mathrm{i}m\varphi) \\ & + \exp(-\mathrm{i}\pi\gamma) \sum_{m=0}^{-\infty} \frac{J_{\gamma-m} - \eta J_{\gamma-m-1}}{H^{(1)}_{\gamma-m} - \eta H^{(1)}_{\gamma-m-1}} \exp(\mathrm{i}m\varphi) \Bigg) \\ f_2 = -\sqrt{\frac{2}{\pi i k}} \Bigg(& \frac{1}{2} \sin \pi\gamma \frac{\exp(\mathrm{i}\varphi/2)}{\sin \varphi/2} + \exp(\mathrm{i}\pi\gamma) \sum_{m=1}^{\infty} \frac{J_{m-\gamma} - \eta J_{m-1-\gamma}}{H^{(1)}_{m-\gamma} - \eta H^{(1)}_{m-1-\gamma}} \exp(\mathrm{i}m\varphi) \\ & + \exp(-\mathrm{i}\pi\gamma) \sum_{m=0}^{-\infty} \frac{J_{\gamma-m} + \eta J_{\gamma+1-m}}{H^{(1)}_{\gamma-m} + \eta H^{(1)}_{\gamma+1-m}} \exp(\mathrm{i}m\varphi) \Bigg). \end{aligned} \tag{3.12}$$

Now consider the limiting cases of these equations.

(a) Zero magnetic flux inside C $(\gamma=0)$:

$$f_1=-\sqrt{\frac{2}{\pi i k}}\sum\frac{J_m+\eta J_{m+1}}{H_m^{(1)}+\eta H_{m+1}^{(1)}}\exp(im\varphi)$$
$$f_2=-\sqrt{\frac{2}{\pi i k}}\sum\frac{J_m-\eta J_{m-1}}{H_m^{(1)}-\eta H_{m-1}^{(1)}}\exp(im\varphi). \tag{3.13}$$

(b) Infinitely thin solenoid ($kR\ll 1$). Contrary to the non-relativistic case, the AB amplitudes are different for $\gamma<\frac{1}{2}$ and $\gamma>\frac{1}{2}$

$$f_1=-\frac{\sin\pi\gamma}{\sqrt{2\pi i k}}\frac{\exp(-i\varphi/2)}{\sin\varphi/2}\qquad f_2=-\frac{\sin\pi\gamma}{\sqrt{2\pi i k}}\frac{\exp(i\varphi/2)}{\sin\varphi/2}\qquad\text{for } 0<\gamma<\tfrac{1}{2}$$
$$f_1=-\frac{\sin\pi\gamma}{\sqrt{2\pi i k}}\frac{\exp(i\varphi/2)}{\sin\varphi/2}\qquad f_2=-\frac{\sin\pi\gamma}{\sqrt{2\pi i k}}\frac{\exp(\frac{3}{2}i\varphi)}{\sin\varphi/2}\qquad\text{for } \tfrac{1}{2}<\gamma<1. \tag{3.14}$$

Nevertheless, the relation between the relativistic and non-relativistic cross sections is the same as before:

$$\sigma_{AB}^{rel}=\sqrt{1-\beta^2}\,\sigma_{AB}^{nonrel}.$$

It is surprising that initial expressions (3.12) for f_1 and f_2 are continuous WRT γ, while f_1 and f_2 given by (3.14) suffer a finite jump at $\gamma=\frac{1}{2}$. The origin of this imaginary controversy becomes clear if we consider the limit of (3.12) for $kR\to 0$ without specifying γ. Retaining in (3.12) only non-vanishing terms, we obtain

$$f_1(kR\to 0,\gamma)=-\sqrt{\frac{2}{\pi i k}}\sin\pi\gamma\left(\frac{1}{2}\frac{\exp(i\varphi/2)}{\sin\varphi/2}-i\eta\frac{1}{(J_{-\gamma}/J_{\gamma-1})\exp(i\pi\gamma)+\eta}\right)$$
$$f_2(kR\to 0,\gamma)=-\sqrt{\frac{2}{\pi i k}}\sin\pi\gamma\left(\frac{1}{2}\frac{\exp(i\varphi/2)}{\sin\varphi/2}+i\eta\frac{\exp(i\varphi)}{(J_{\gamma-1}/J_{-\gamma})\exp(-i\pi\gamma)+\eta}\right).$$

Now we take into account that for $kR\to 0$

$$\frac{J_{-\gamma}}{J_{\gamma-1}}\approx\left(\frac{kR}{2}\right)^{1-2\gamma}\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}.$$

It is convenient to present the first factor of this equation in the form

$$\left(\frac{kR}{2}\right)^{1-2\gamma}=\exp\left((1-2\gamma)\ln\frac{kR}{2}\right).$$

If kR is small enough that $|(1-2\gamma)\ln(kR/2)|\gg 1$, then

$$\left(\frac{kR}{2}\right)^{1-2\gamma}=\begin{cases}0 & \text{for } 0<\gamma<\frac{1}{2}\\ \infty & \text{for } \frac{1}{2}<\gamma<1\end{cases}$$

and we arrive at (3.14). On the other hand, if γ is so close to $\frac{1}{2}$ that $|(1-2\gamma)\ln(kR/2)|\ll 1$, then

$$\left(\frac{kR}{2}\right)^{1-2\gamma}\approx 1+(1-2\gamma)\ln\frac{kR}{2}.$$

Substituting this into f_1 and f_2, we obtain

$$f_1(\gamma=\tfrac{1}{2},kR\to 0)=-\sqrt{\frac{2}{\pi i k}}\left(\frac{1}{2}\frac{\exp(i\varphi/2)}{\sin\varphi/2}-\frac{\eta}{1-i\eta}\right)$$
$$f_2(\gamma=\tfrac{1}{2},kR\to 0)=-\sqrt{\frac{2}{\pi i k}}\left(\frac{1}{2}\frac{\exp(i\varphi/2)}{\sin\varphi/2}-\frac{\eta}{1+i\eta}\exp(i\varphi)\right). \tag{3.15a}$$

Equations (3.14) and (3.15) correspond to different physical situations. The fact that different orders of limiting procedures may lead to different physics is not new. Particularly, the recent discussion on the validity of the first Born approximation for the description of the AB effect (see, e.g. [14]) is due to this fact. In figure 8, the scattering cross sections corresponding to the exact relativistic impenetrability BC (3.11) are given. Comparing them with the approximately relativistic cross sections presented in figure 7, we observe the perfect agreement between them in spite of the sharp difference between the corresponding scattering amplitudes (3.8) and (3.12). This remains an enigma for us. We summarize briefly the content of this section. We obtain approximate ((3.8), (3.9)) and exact ((3.12)-(3.14)) relativistic scattering amplitudes. They correspond to the non-relativistic (3.7) and relativistic (3.11) impenetrability conditions, respectively.

3.2. Relativistic scattering on the impenetrable sphere

Let an incoming wave propagate in the positive z direction with positive energy and helicity. Then,

$$\psi_{\mathrm{inc}} = \exp(\mathrm{i}kz)\begin{bmatrix}1\\0\\\eta\\0\end{bmatrix}$$

$$\eta = \left(\frac{\mathscr{E}-\mu c^2}{\mathscr{E}+\mu c^2}\right)^{1/2}$$

$$k = \frac{1}{\hbar c}(\mathscr{E}^2-\mu^2c^4)^{1/2}.$$

Let this incoming wave be scattered by the impenetrable sphere S of radius R. The complete wavefunction is $\psi = \psi_{\mathrm{inc}} + \psi_{\mathrm{S}}$. As $r \to \infty$

$$\psi_{\mathrm{S}} = \frac{1}{r}\exp(\mathrm{i}kr)f \qquad f = \begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix}.$$

The components f_i of the spinorial scattering amplitude are determined by the particular choice of the impenetrability condition. For instance, we may require the large components of the Dirac wavefunction to vanish at $r = R$. Then,

$$\begin{aligned}
f_1 &= \frac{\mathrm{i}}{k}\sum(2l+1)\frac{J_{l+1/2}}{H^{(1)}_{l+1/2}}P_l(\cos\theta) \qquad f_2 = 0\\
f_3 &= \frac{\mathrm{i}\eta}{k}\sum\left(l\frac{J_{l-1/2}}{H^{(1)}_{l-1/2}} + (l+1)\frac{J_{l+3/2}}{H^{(1)}_{l+3/2}}\right)P_l(\cos\theta)\\
f_4 &= \frac{\mathrm{i}\eta}{k}\exp(\mathrm{i}\varphi)\sum\left(\frac{J_{l-1/2}}{H^{(1)}_{l-1/2}} - \frac{J_{l+3/2}}{H^{(1)}_{l+3/2}}\right)P^1_l(\cos\theta)
\end{aligned} \tag{3.15b}$$

where $P^m_l(\cos\theta)$ are the Legendre functions of the first kind. It turns out that small components of the Dirac wavefunction are of order $(kR)^{-2}$ at $r = R$. The other possible choice is the disappearance of the Dirac probability current at $r = R$. This is fulfilled if

$$\mathrm{i}\boldsymbol{\alpha n}\psi = \beta\psi \tag{3.16}$$

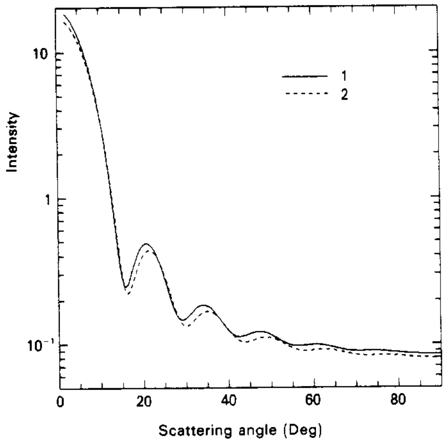

Figure 9. The relativistic intensity of electron scattering on the impenetrable sphere S of radius R. Curves 1 and 2 correspond to the vanishing of large components of the Dirac wavefunction at the surface of S and to the impenetrability BC (3.16), respectively.

Here $\boldsymbol{n}$ is the normal to the surface of S

$$\boldsymbol{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta, 0).$$

The following components of the spinorial amplitude correspond to this impenetrability condition:

$$\begin{aligned} f_1 &= \frac{\mathrm{i}}{k}\sum[(l+1)C_{l+1} + l\,\mathrm{d}_l]P_l(\cos\theta) \\ f_2 &= -\frac{\mathrm{i}}{k}\exp(\mathrm{i}\varphi)\sum(C_{l+1} - \mathrm{d}_l)P_l^1(\cos\theta) \\ f_3 &= \frac{\mathrm{i}\eta}{k}\sum[lC_l + (l+1)d_{l+1}]P_l(\cos\theta) \\ f_4 &= \frac{\mathrm{i}\eta}{k}\exp(\mathrm{i}\varphi)\sum(C_l - d_{l+1})P_l^1(\cos\theta). \end{aligned} \tag{3.17}$$

Here

$$C_l = \frac{J_{l-1/2} + \eta J_{l+1/2}}{H^{(1)}_{l-1/2} + \eta H^{(1)}_{l+1/2}} \qquad d_l = \frac{J_{l+1/2} - \eta J_{l-1/2}}{H^{(1)}_{l+1/2} - \eta H^{(1)}_{l-1/2}}.$$

For either of these impenetrability conditions the cross section is given by

$$\sigma(\theta) = \frac{1}{1+\eta^2}\sum|f_i|^2.$$

The corresponding angular dependences are shown in figure 9. As for the case of the infinite cylinder, we observe an excellent agreement between cross sections corresponding to quite different impenetrability BC (in spite of the sharp difference between the scattering amplitudes (3.15) and (3.17)).

4. Discussion

We have seen that there is a variety of theoretical possibilities for preventing the trapping of incoming particles into the excluded region. This is bad. Fortunately, only few of them could be easily realized in practice. In a non-relativistic case the most promising seems to be the BC $\psi = 0$ at the boundary of the inaccessible region. In this case, the normal component of the probability current $\boldsymbol{j}$ (see section 2) vanishes on that boundary for any value of the VP $\boldsymbol{A}$. This BC is easy to realize experimentally: it is enough to switch on the large repulsion inside the inaccessible region where $H \neq 0$. As an illustration of the complications arising from another choice of the impenetrability condition, consider how one could verify experimentally the non-existence of the AB effect in simply connected (in the physical sense) regions of space. To do this, we may study, e.g., the charge particle diffraction on the impenetrable sphere S with and without a toroidal solenoid inside it. Let the impenetrability condition in the absence of the magnetic field be chosen as $\partial\psi_0/\partial r = 0$ at $r = R$. Now insert the toroidal solenoid inside S. As the radial component of the VP does not now vanish outside S, it is impossible to vanish the normal component of $\boldsymbol{j}$ (see (2.16)) using the same BC $\partial\psi/\partial r = 0$ for $r = R$ (moreover, if this BC were nevertheless realized, the inevitable shift of the diffraction pattern should take place due to particle penetration inside the sphere S where the toroidal solenoid is situated). The transformed impenetrability condition (satisfying $j_r = 0$ at $r = R$) should be of the form (this follows from (A5.2))

$$\psi|_{r=R} = \psi_0 \exp\left(\frac{\mathrm{i}e}{\hbar c}\frac{\partial\alpha}{\partial r}\right)\bigg|_{r=R}$$

where the function α is defined by (A3.1) and ψ_0 is the solution of the free Schrödinger equation with the boundary condition $\partial\psi_0/\partial r = 0$ at $r = R$. It is not clear how this rather complicated BC can be realized experimentally.

We conclude: the BC $\psi = 0$ imposed at the boundary of the excluded region has universal meaning in the non-relativistic case and does not lead to experimental complications. On the other hand there are no solutions of the Dirac equation meeting the $\psi = 0$ condition at the frontier of the inaccessible region. This means that the $\psi = 0$ BC partly loses its physical sense and uniqueness in the relativistic case. The BC (3.7) (which leads to the disappearance of the large components of the Dirac wavefunction and to the non-vanishing at the normal component of the relativistic probability current) is easy to realize experimentally by creating the large repulsive barrier. In theory, the truly relativistic BC (3.10) or (3.11) seems to be more promising (for which the normal component of the same current vanishes), but it is not clear how to prepare them in practice. A good numerical coincidence of the cross sections corresponding to BC (3.7) and (3.11) observed in the previous section removes this insufficiency. In [15] the electron scattering on the impenetrable toroidal solenoid was studied. As the impenetrability was achieved by imposing the $\psi = 0$ BC on its surface, these calculations can be useful in analysing the existing experimental data [12].

Appendix 1

The authors of [7] consider the toroidal cavity $T(a_1 < \rho < a_2, |z| < b_1)$ with non-vanishing magnetic field inside it

$$H_\rho = H_r = 0 \qquad H_\varphi = \frac{\phi}{2b_1(a_2 - a_1)} \tag{A1.1}$$

(ϕ is the magnetic field flux). Outside T the magnetic field is zero everywhere. Furthermore, two different VP are presented in [7], which give the same magnetic field

(A1.1). They are equal to zero for $|z|>b_1$. For $|z|<b_1$ they are defined as

$$A_z^{(1)}=\begin{cases}\dfrac{\phi}{2b_1} & \text{for } \rho<a_1\\[2ex] \dfrac{\phi}{2b_1}\dfrac{a_2-\rho}{a_2-a_1} & \text{for } a_1<\rho<a_2\\[2ex] 0 & \text{for } \rho>a_2\end{cases} \tag{A1.2}$$

$$A_z^{(2)}=\begin{cases}0 & \text{for } \rho<a_1\\[2ex] -\dfrac{\phi}{2b_1}\dfrac{\rho-a_1}{a_2-a_1} & \text{for } a_1<\rho<a_2\\[2ex] -\dfrac{\phi}{2b_1} & \text{for } \rho>a_2.\end{cases} \tag{A1.3}$$

Let the charge particle be confined to the cylindrical cavity C ($|z|<b$ $(b<b_1)$, $\rho<a$ $(a<a_1)$). Now impose on the wavefunction the Dirichlet BC in the radial direction ($\psi=0$ for $\rho=a$) and periodic BC in the z direction

$$\psi(z=b)=\psi(z=-b). \tag{A1.4}$$

The solutions of the Schrödinger equation (SE)

$$-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}+\left(\frac{\partial}{\partial z}-\frac{ie}{\hbar c}A_z^{(1,2)}\right)^2\psi\right]=E\psi \tag{A1.5}$$

with the chosen BC of the form

$$\psi_{\mathrm{nsm}}^{(1,2)}\sim\exp\left(\frac{i\pi nz}{b}\right)\exp(im\varphi)J_m(\Lambda_{ms}\rho/R). \tag{A1.6}$$

Here Λ_{ms} means the Sth non-vanishing zero of the Bessel function $J_m(x)$. To these wavefunctions correspond eigenvalues

$$\begin{aligned} E_{\mathrm{nsm}}^{(1)}&=\frac{\hbar^2}{2\mu}\left[\frac{\Lambda_{ms}^2}{a^2}+\left(\frac{n\pi}{b}-\frac{e\phi}{\hbar c}\frac{1}{2b_1}\right)^2\right] && \text{for } \boldsymbol{A}=\boldsymbol{A}_1\\ E_{\mathrm{nsm}}^{(2)}&=\frac{\hbar^2}{2\mu}\left[\frac{\Lambda_{ms}^2}{a^2}+\left(\frac{n\pi}{b}\right)^2\right] && \text{for } \boldsymbol{A}=\boldsymbol{A}_2.\end{aligned} \tag{A1.7}$$

Clearly, $E_{\mathrm{nsm}}^{(1)}\neq E_{\mathrm{nsm}}^{(2)}$, i.e. eigenvalues of the SE in the simply connected region with $H=0$ depend on the particular choice of the VP. This seems strange, as it is invariant relative to the gauge transformation

$$\boldsymbol{A}_1=\boldsymbol{A}_2+\operatorname{grad}\chi \qquad \psi_1=\tilde{\psi}_2\exp\left(\frac{ie\chi}{\hbar c}\right) \tag{A1.8}$$

where the function χ equals $\phi z/2b_1$ for $|z|<b_1$, $x=\frac{1}{2}\phi$ for $z>b_1$ and $-\frac{1}{2}\phi$ for $z<-b_1$. Let ψ_1 satisfy the same periodic BC (A1.4). It follows from (A1.8) that $\tilde{\psi}_2$ does not meet this BC

$$\tilde{\psi}_2(z=-b)=\psi_2(z=b)\exp\left(\frac{ie\phi}{\hbar c}\right)\neq\tilde{\psi}_2(z=b).$$

Now the difference between $E^{(1)}$ and $E^{(2)}$ becomes understandable as the corresponding wavefunctions ψ_1 and ψ_2 are not related by a gauge transformation. In other words, the BC (A1.4) is not gauge invariant. Although we can impose this BC in one particular gauge, we cannot require its fulfillment in the other gauge if we wish to stay in the framework of a gauge invariant theory. The gauge invariance of the SE is not enough for the absence of observable effects. The BC should be gauge invariant too.

Appendix 2.

After this digression we return to subsection 2.3. Outside the solenoid the VP $\boldsymbol{A}$ may be presented as a gradient of some function χ. As $\oint \boldsymbol{A}e\,\mathrm{d}l \neq 0$ for the closed contour passing through the solenoid hole, this function turns out not to be single valued (more accurately: discontinuous). In exact terms for the toroidal solenoid $(\rho - d|^2 + z^2 = R_0^2$ the function χ undergoes a jump from $\phi/2$ to $-\phi/2$ when one crosses the circle of radius $d-R$ lying in the $z=0$ plane (i.e. in the equatorial plane of the solenoid). At large distances it falls as r^{-2}: $\chi \approx -(\pi g/4r^2)\, \mathrm{d}R_0^2 \cos\theta (g = (\phi/2\pi)(d-\sqrt{d^2-R_0^2})^{-1})$. The explicit expression for χ may be found in [16]. The unitary transformation

$$\boldsymbol{A} = \boldsymbol{A}' + \operatorname{grad}\chi \qquad \psi = \psi' \exp\left(\frac{\mathrm{i}e\chi}{\hbar c}\right) \tag{A2.1}$$

may be used to eliminate the VP outside the solenoid. As ψ is assumed to be single valued, ψ' is a non-single valued function satisfying the free SE and having discontinuities at the same place as χ

$$\psi'(\rho \leqslant d - R_0, z = 0-) = \exp\left(\frac{-\mathrm{i}e\phi}{\hbar c}\right) \psi'(\rho \leqslant d - R_0, z = 0+). \tag{A2.2}$$

However, to find the non-single valued solution of the free Schrödinger equation is no easier than getting single valued solutions of the initial unabridged SE. Some profit may be obtained from (A2.1) when $\psi = \psi' = 0$ at the discontinuity region of the χ function. This can be achieved, for instance, by switching on the infinite repulsion inside the sphere S. In this case, the discontinuity condition (A2.2) becomes trivial $(0=0)$ while (A2.1) turns out to be the unitary transformation between the single valued wavefunction with $\boldsymbol{A} \neq 0$ and $\boldsymbol{A} = 0$. As a result, the non-vanishing of the VP outside S does not lead to observable consequences. It does not follow from (2.16) that $\psi = 0$ inside the sphere S; so the transformation (A2.1) is useless.

Appendix 3.

Fortunately, there exists another VP $\boldsymbol{A}'$ [16, 17] which differ from zero in the immediate neighbourhood of the toroidal solenoid. This VP has the single non-vanishing component which equals $A'_z = g\ln(d + \sqrt{R_0^2 - z^2/\rho})$ inside the solenoid. Outside it $A'_z \neq 0$ only in the region $|z| < R_0$, $0 \leqslant \rho \leqslant d - \sqrt{R_0^2 - z^2}$ broken in figure 10) where it equals $g\ln[(d + \sqrt{R_0^2 - z^2})/(d - \sqrt{R_0^2 - z^2})]$. An analogue of $\boldsymbol{A}'$ for the infinite cylindrical solenoid is also known [17]. Its properties have recently been discussed in this journal [18]. The VP $\boldsymbol{A}$ and $\boldsymbol{A}'$ are connected via a single valued gauge transformation. We prove this fact for the infinitely thin solenoid. In this case, A'_z reduces to [19]

On the other hand, the cartesian components of $\boldsymbol{A}$ are expressible in the form [16, 19]

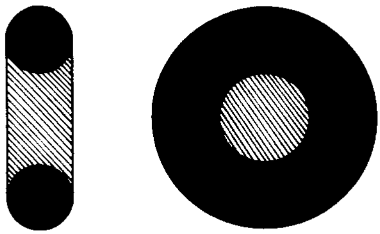

Figure 10. The vector potential of the toroidal solenoid in a non-Coulomb gauge. Outside the solenoid VP differs from zero in the dashed region only.

$A'_z = \phi\delta(z)\theta(d-\rho)$.

(for the thin solenoid)

$$A_x = \frac{\partial^2\alpha}{\partial x\partial z} \qquad A_y = \frac{\partial^2\alpha}{\partial y\partial z} \qquad A_z = -\frac{\partial^2\alpha}{\partial x^2} - \frac{\partial^2\alpha}{\partial y^2}$$
$$\alpha = \frac{\phi}{4\pi}\int\int \frac{\mathrm{d}x_1\,\mathrm{d}y_1}{|\boldsymbol{r}-\boldsymbol{r}_1|}. \tag{A3.1}$$

(Here integration is performed over the circle of radius d lying in the plane $z=0$). The A_z component may be presented in a slightly different form $A_z = \partial^2\alpha/\partial z^2 - \Delta\alpha$. Bearing in mind that $\Delta(1/|\boldsymbol{r}-\boldsymbol{r}_1|) = -4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}_1)$ we get $\Delta\alpha = -\phi\delta(z)\theta(d-\rho) = -A'_z$

$$\boldsymbol{A} = \boldsymbol{A}' + \mathrm{grad}\,\frac{\partial\alpha}{\partial z}. \tag{A3.2}$$

This equation is just the desired transformation between $\boldsymbol{A}$ and $\boldsymbol{A}'$.

Appendix 4.

We now discuss the properties of the α function. The double integral occurring in the definition of α (A3.1) may be expressed through the linear integrals [16, 20]

$$\int\int \frac{\mathrm{d}\rho_1\,\mathrm{d}z_1}{|\boldsymbol{r}-\boldsymbol{r}_1|} = \frac{2}{\sqrt{d}}\int_0^d \mathrm{d}x\,\sqrt{x}\,Q_{-1/2}\left(\frac{r^2+x^2}{2\rho x}\right)$$
$$= 2\pi(\sqrt{z^2+d^2}-|z|) - 2\sqrt{d}\int_0^\rho \frac{\mathrm{d}x}{\sqrt{x}}\,Q_{1/2}\left(\frac{z^2+d^2+x^2}{2\,\mathrm{d}x}\right). \tag{A4.1}$$

From the first line of this expression it follows that for $\rho > d$ the argument $y(=\rho^2+z^2+x^2/2\rho x)$ of the Legendre function $Q_{-1/2}$ always exceeds 1 for all x in the interval $0 \leqslant x \leqslant d$. This means (as the cut of the Legendre functions coincides with the interval $(-1, 1)$) that the function α and all its derivatives are continuous functions of z for $\rho > d$. For $\rho < d$, y acquires the value 1 for $z=0$, $x=\rho$. In this case the function α and its derivatives may possess singularities. This explicitly demonstrates the second line of (A4.1). In fact the argument $z^2+x^2+d^2/2\,\mathrm{d}x$ of the $Q_{1/2}$ always exceeds 1 for all $\rho < d$. Thus for $\rho < d$ all the singularities of the α function are due to the first term of this line: $(\phi/2)(\sqrt{z^2+d^2}-|z|)$. The first z derivative of this expression has a finite jump for $z=0$ (equal to $-\phi$), while the second z derivative has a δ-type singularity $(-\phi\delta(z))$. We know that for $\rho>d$ the function α and its derivatives have no singularities as a function of z. It follows from this that for $\rho>d$ the singularities of the first and second terms in the second line of (A4.1) mutually compensate each other.

Appendix 5.

The singularities of the transformed wavefunction

$$\psi'=\psi\exp\left(-\frac{\mathrm{i}e}{\hbar c}\frac{\partial\alpha}{\partial z}\right) \tag{A5.1}$$

lie in the equatorial plane $(z=0, \rho<d)$ of the solenoid, which is completely inside the sphere S. As the gauge transformation (A3.2) eliminates the VP outside the sphere S, ψ' satisfies the free Schrödinger equation with BC

$$\bar{\psi}'\frac{\partial\psi'}{\partial r}-\psi'\frac{\partial\bar{\psi}'}{\partial r}=0 \qquad \text{at } r=R.$$

Thus, it coincides with the wavefunction ψ_0 defined by (2.14). After this identification we may return to the original wavefunction using the transformation inverse to (A5.1)

$$\psi=\psi_0\exp\left(\frac{\mathrm{i}e}{\hbar c}\frac{\partial\alpha}{\partial z}\right). \tag{A5.2}$$

As for $r\to\infty$, $\partial\alpha/\partial z\approx-(\phi d^2/4r^2)\cos\theta$ the transformation (A5.2) does not change the asymptotic behaviour of the wavefunction, and as a consequence, leads to the same scattering amplitude and cross section. We conclude: the impenetrability condition (2.16) guarantees the absence of observable effects arising from the solenoid presence inside the sphere S.

Appendix 6.

Let the BC

$$\psi_0(r=R)=f_0(\theta,\varphi) \tag{A6.1}$$

be imposed on the surface of the sphere S. Here f_0 is an arbitrary single valued function. The space region outside S is a simply connected one. Insert into S the toroidal solenoid. As a result, there appears the VP $\boldsymbol{A}$ outside the sphere S. Let, in the presence of a solenoid inside S, the same BC (A6.1) be fulfilled on the surface of S. The gauge transformation (A3.2) may be used to eliminate VP outside S. According to (A5.1), the transformed wavefunction ψ_0' satisfies the free SE with BC

$$\psi_0'(r=R)=f_0(\theta,\varphi)\exp\left(-\frac{\mathrm{i}e}{\hbar c}\frac{\partial\alpha}{\partial z}\right)\bigg|_{r=R}. \tag{A6.2}$$

The free wavefunctions ψ_0 and ψ_0' are physically different and this could lead to observable consequences.

The presence of a solenoid inside S becomes unobservable in two cases. First, when the BC on the surface of S explicitly depends on the magnetic flux inside the solenoid. This magnetic flux being equal to ϕ, the above BC should have the form

$$\psi(r=R)=f_0 \exp\left(\frac{ie}{\hbar c}\frac{\partial\alpha}{\partial z}\right)\Bigg|_{r=R}. \tag{A6.3}$$

Applying to wavefunction ψ the gauge transformation (A3.2), we arrive at the BC (A6.1) which implies the unobservability of the solenoid inside S. The second case takes place when the gauge invariant BC is imposed on the surface of S. The quantum probability current being the gauge invariant quantity, the disappearance of its normal component at the surface of S is a gauge invariant BC as well. The story is not complete. The reason is that on the surface of S we should impose the concrete realization of the impenetrability conditions (2.10) or (2.16). These realizations, being connected by the gauge transformation (A5.2), are in one-to-one correspondence with each other. It may happen that for $A \neq 0$ and $A=0$ those particular impenetrability BC are chosen which are not connected by a gauge transformation. These situations are physically different and this fact could in principle be verified experimentally (e.g., by the scattering of charged particles). The only exception is the $\psi=0$ BC. In this case, the BC (2.16) is satisfied for any value of VP $\boldsymbol{A}$.

We briefly summarize the content of this rather lengthy appendix. It turns out that the simply connected nature of the regions of space with $H=0$ does not guarantee the non-observability of physical effects originating from non-vanishing of $\boldsymbol{A}$ in a simply connected region (in spite of the possibility of eliminating $\boldsymbol{A}$ by means of a well behaved gauge transformation).

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